Int. J. Solids Structures, 1966, Vol. 2, pp. 105 to 124. Pergamon Press Ltd. Printed in Great Britain

BROADBAND RANDOM VIBRATIONS OF ELASTIC SYSTEMS

V. V. BOLOTIN

Moscow Energetic Institute, Moscow, USSR

Abstract—Considered are vibrations of elastic systems under the action of external forces which represent a space–time broadband random process. On the basis of the asymptotic method, advanced by the author, and of estimates for the density of natural frequencies of shells, integral estimates for displacements and stresses are proposed which occur in elastic systems. At the basis of the method advanced there lies the replacement of summation over separate natural modes of free vibrations by integration over a certain range of frequencies (or wave numbers). The question concerning limitations of this method is discussed. Investigated is the influence of the characteristics of the loading of the type of damping as well as of the type of boundary conditions on mean squares and spectral densities of displacements and stresses.

INTRODUCTION

IN THIS PAPER, the problems of vibration of elastic systems (beams, plates and shells) under random forces represented as a broadband space–time process are considered. As an example the random vibrations of structures due to acoustic pressure fluctuations radiated from the hot jet may be mentioned. It is well known [1] that the frequencies of the acoustic pulsations are distributed practically over the whole sound range. The pulsations of acoustic pressure can excite a great number of structural modes simultaneously. Thus the common methods based on the expansion of the solutions into a series of natural modes are inconvenient. At the same time the possibility arises of obtaining approximate solutions based on asymptotic properties.

The method based on this idea was proposed by the author several years ago [2–4]. Two features of eigenvalue spectrum are employed in this method. First, the asymptotic behaviour of the highest eigenfrequencies and associated modes, and, secondly, the sufficiently great density of eigenfrequencies of elastic plates, shells and other elastic bodies. The method consists essentially in replacing summation of the contribution of each mode by integration over a certain region in wave numbers space. This is analogous to the passage from the quantum statistics to the classical one. The wider the excitation spectrum and the higher the density of natural frequencies, the better is the approximation by the asymptotic method. Unlike the direct summation method, the integral method proposed, yields in many cases the results in closed form. Thus we have the possibility to study systematically the influence of different factors on the random behaviour of elastic systems.

The article is written according to the following plan. In Section 1 brief information on the asymptotic method is presented [2, 3]. The principle of replacing the summation over vibration modes by integration over the wave numbers space is discussed in Section 2. Examples of application of this method to thin elastic plates are given in Sections 3 and 4. In these sections the influence of spectral and correlation characteristics of loading, the influence of boundary conditions and of the type of damping on mean square displacements and mean square stresses are investigated. Section 5 is devoted to the consideration of the shear and rotationary inertia effects. In this section the boundaries of applicability of classical plate theory to the problem of broadband vibration are discussed. In Section 6 examples of the application of the method to thin elastic shells are given. The evaluation of mean square stresses and strains in the case of thin spherical shells is discussed in particular. In the last section an application of the proposed method to the estimation of spectral densities, effective frequencies and other parameters is given.

This paper is based on material presented at the 11th International Congress of Applied Mechanics, Munich, 1964. A short abstract of this report is to be published in the Proceedings of the Congress [5].

1. THE ASYMPTOTIC METHOD IN THE THEORY OF VIBRATION OF ELASTIC BODIES

Referring for details to papers [2, 3], let us present some information on the asymptotic method for the investigation of natural frequencies and modes of elastic systems for sufficiently high wave numbers.

According to this method the asymptotic solution for the natural mode is expressed as a sum of the internal solution and correction solutions which are called the dynamic boundary effects. Each expression satisfies the differential equations and the conditions on one of the boundaries of the rectangular region. The number of these expressions is equal to the number of boundaries. Then the obtained expressions must be matched. This procedure is similar to the matching of the membrane solution and the boundary effects in the theory of shells or to the matching of the viscous and inviscid solutions in hydrodynamics. The matching can be achieved only approximately. The faster the boundary effects decay, the less is the error of the asymptotic solution. As a result the equations can be obtained for the parameters defining the internal solution and the boundary effects. Then the asymptotic expression for natural modes can be obtained for the whole region except corners and ribs. This is typical for all methods which use the concept of the boundary effect.

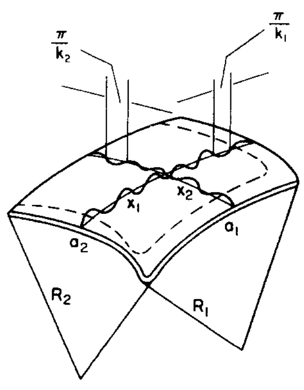

FIG. 1. Schematics of the shell.

As an example consider a thin elastic shell with constant thickness h, referred to orthogonal coordinates x_1, x_2, which coincide with curvature lines (Fig. 1). Let $\varkappa_1$ and $\varkappa_2$ be the principal curvatures of the middle surface, E the elastic modulus, ρ the material density, D the cylindrical rigidity, w the normal displacement, ψ the stress function multiplied by h, ω the natural frequency. If the characteristic wavelength is sufficiently small compared with $\varkappa_1^{-1}$ and $\varkappa_2^{-1}$, the equations for natural modes can be taken in the form

$$D\Delta\Delta w-\left(\varkappa_2\frac{\partial^2\psi}{\partial x_1^2}+\varkappa_1\frac{\partial^2\psi}{\partial x_2^2}\right)-\rho h\omega^2 w=0,$$
$$\frac{1}{Eh}\Delta\Delta w+\left(\varkappa_2\frac{\partial^2 w}{\partial x_1^2}+\varkappa_1\frac{\partial^2 w}{\partial x_2^2}\right)=0. \tag{1.1}$$

The asymptotic solution of the equations (1.1) for the rectangular (in a general sense) region with sides a_1 and a_2 and with constant middle surface metrics have been obtained in paper [2]. In particular, for natural frequencies the formula

$$\omega^2\sim\frac{D}{\rho h}\left[(k_1^2+k_2^2)^2+\frac{Eh\varkappa_1^2}{D}\frac{(k_1^2\chi+k_2^2)^2}{(k_1^2+k_2^2)^2}\right] \tag{1.2}$$

was given, where $\chi=\varkappa_2/\varkappa_1$ ($\varkappa_1\neq 0$). The wave numbers k_1 and k_2 are calculated from the following transcendental equations:

$$k_1a_1=\operatorname{arc\,tg}u_{11}(k_1,k_2)+\operatorname{arc\,tg}u_{12}(k_1,k_2)+m_1\pi,$$
$$k_2a_2=\operatorname{arc\,tg}u_{21}(k_1,k_2)+\operatorname{arc\,tg}u_{22}(k_1,k_2)+m_2\pi \tag{1.3}$$
$$(m_1,m_2=1,2,\ldots).$$

The functions $u_{\alpha\beta}(k_1,k_2)$ depend on the boundary conditions. For free supported edges all functions $u_{\alpha\beta}(k_1,k_2)$ are equal to zero; the asymptotic solution in this case coincides with the well known exact solution.

The asymptotic solution is valid in the whole domain of sufficiently high wave numbers except in the regions of degeneration of the dynamic boundary effect. The details may be found in article [2]. For example, the dynamic boundary effect never degenerates for plates and spherical shells. For the cylindrical shell the boundary effect degenerates only for sufficiently small wave numbers:

$$k_1^2+k_2^2\leq\left(\frac{Eh\varkappa_1^2}{D}\right)^{\frac{1}{2}}.$$

As a most simple example let us consider the application of this method to the problem of natural vibrations of a rectangular clamped plate with sides a_1 and a_2. The equations (1.3) in this case take on the form

$$k_1a_1=2\operatorname{arc\,tg}\frac{k_1}{(k_1^2+2k_2^2)^{\frac{1}{2}}}+m_1\pi,$$
$$k_2a_2=2\operatorname{arc\,tg}\frac{k_2}{(k_2^2+2k_1^2)^{\frac{1}{2}}}+m_2\pi\quad(m_1,m_2=1,2,\ldots). \tag{1.4}$$

Table 1

m_1	m_2	$\frac{k_1 a_1}{\pi}$	$\frac{k_2 a_2}{\pi}$	$\bar{\omega}$ Proposed method	$\bar{\omega}$ Calculations of Iguchi	$\Delta\bar{\omega}$ (%)
1	1	4/3	4/3	3·556	3·646	2·53
1	2	1·2027	2·4372	7·386	7·437	0·69
2	2	7/3	7/3	10·889	10·965	0·70
1	3	1·1420	3·4688	13·337	13·398	0·42
2	3	2·2556	3·4012	16·656	16·717	0·37
3	3	10/3	10/3	22·222	—	—
1	4	1·1084	4·4816	21·313	—	—
2	4	2·2038	4·4366	24·540	24·631	0·36
3	4	3·2784	4·3832	29·960	—	—
4	4	13/3	13/3	37·556	—	—

Here the functions $\operatorname{arc\,tg} u_{\alpha\beta}$ are taken in the sense of their principal values. The natural vibration frequencies are determined by formula

$$\omega \sim (k_1^2 + k_2^2)\left(\frac{D}{\rho h}\right)^{\frac{1}{2}}. \tag{1.5}$$

A comparison of this result with the very reliable results of Iguchi [6], which have been obtained by means of a variational method, is presented in Table 1. The calculations were performed for a square plate with sides $a_1 = a_2 = a$. Here $\bar{\omega}$ is the dimensionless frequency coefficient in the formula

$$\omega = \frac{\pi^2 \bar{\omega}}{a^2}\left(\frac{D}{\rho h}\right)^{\frac{1}{2}}.$$

As is seen from the table even for the lower frequency the deviation does not exceed 3 per cent. This deviation decays rapidly with an increase of the frequency.

The asymptotic expression for natural modes can be easily obtained. Thus, near the boundary $x_1 = 0$ (except in the neighbourhood of corner points) the expression can be obtained

$$\begin{aligned} w(x_1, x_2) \sim \frac{(k_1^2 + 2k_2^2)^{\frac{1}{2}}}{2^{\frac{1}{2}}(k_1^2 + k_2^2)^{\frac{1}{2}}} \Bigg\{ &\sin k_1 x_1 - \frac{k_1 \cos k_1 x_1}{(k_1^2 + 2k_2^2)^{\frac{1}{2}}} \\ &+ \frac{k_1}{(k_1^2 + 2k_2^2)^{\frac{1}{2}}} \exp[-x_1 (k_1^2 + 2k_2^2)^{\frac{1}{2}}] \Bigg\} \sin k_2 (x_2 - x_2^0). \end{aligned} \tag{1.6}$$

As x_1 increases the boundary effect rapidly decays and the expression (1.6) approaches the internal solution

$$w^*(x_1, x_2) = \sin k_1 (x_1 - x_1^0) \sin k_2 (x_2 - x_2^0). \tag{1.7}$$

Here x_1^0 and x_2^0 are the limiting phases (certain constants depending on the boundary conditions).

Note that from the formulae (1.3) the rougher approximation may be obtained:

$$k_1 a_1 = m_1 \pi + \mathcal{O}(1), \qquad k_2 a_2 = m_2 \pi + \mathcal{O}(1) \qquad (m_1, m_2 = 1, 2, \ldots). \tag{1.8}$$

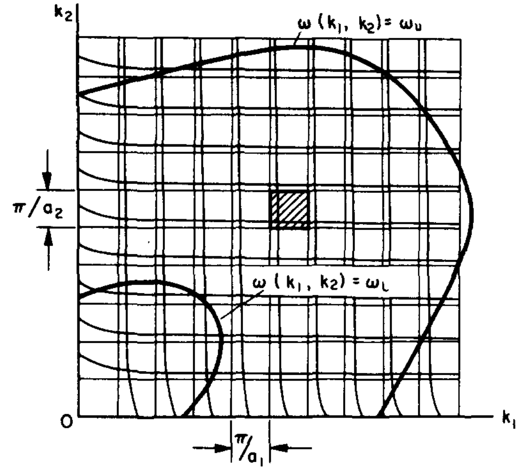

FIG. 2. Family of curves, determined by equation (1.7), in the plane of wave numbers.

These formulae are analogous to the well-known Courant's asymptotic estimates [7] for the eigenvalues in the vibration problem of membranes and thin plates. It is essential that for shells these asymptotic estimates are valid only if the boundary effect does not degenerate. This is clear from the physical point of view: the degeneration of the boundary effect indicates a strong influence of the boundary conditions on the mode behaviour in the internal region.

On Fig. 2 two sets of curves corresponding to the integer numbers m_1 and m_2 in equations (1.8) are shown. The wave numbers are defined as coordinates of the intersection points. If the shell is simply supported all $u_{\alpha\beta}$ are equal to zero and we obtain two sets of straight lines parallel to coordinate axes. The dimensions of cells formed by these lines are $\Delta k_1 = \pi/a_1$ and $\Delta k_2 = \pi/a_2$. In the general case a variation of the boundary conditions cannot displace the intersection points by more than Δk_1 and Δk_2. This follows from formula (1.8).

2. THE METHOD FOR OBTAINING ASYMPTOTIC ESTIMATES FOR MEAN SQUARES OF DISPLACEMENTS AND STRESSES

Let us study now the vibrations of an elastic system under the action of random loads. The usual procedure for solving this problem is based on the expansion of displacements, stresses, etc. in series of principal modes. Thus, the partial differential equations of the problem are reduced to an infinite number of ordinary differential equations. Further, this system must be truncated and standard methods of statistical dynamics used. As an example let us consider a two-dimensional elastic system, characterized by the displacement function $w(x_1, x_2, t)$. Let us assume

$$w(x_1, x_2, t) = \sum_{\alpha} f_\alpha(t)\varphi_\alpha(x_1, x_2), \tag{2.1}$$

where $\varphi_\alpha(x_1, x_2)$ are the natural modes, $f_\alpha(t)$ are the generalized coordinates. Introducing some additional assumptions relating to the character of the dissipative forces we obtain

$$\frac{d^2 f_\alpha}{dt^2} + 2\beta_\alpha \omega_\alpha \frac{df_\alpha}{dt} + \omega_\alpha^2 f_2 = Q_\alpha(t) \quad (\alpha = 1, 2, \ldots). \tag{2.2}$$

Here ω_α are the natural vibration frequencies of the nondissipative system, β_α are the damping coefficients, $Q_\alpha(t)$ are the generalized forces. The generalized forces are connected with the loading density $q(x_1, x_2, t)$, natural modes $\varphi_\alpha(x_1, x_2)$ and surface mass density ρh by the expression

$$Q_\alpha(t) = \frac{\iint q(x_1, x_2, t)\varphi_\alpha(x_1, x_2)\, dx_1\, dx_2}{\iint \rho h \varphi_\alpha^2(x_1, x_2)\, dx_1\, dx_2}. \tag{2.3}$$

Let the load intensity $q(x_1, x_2, t)$ be the stationary ergodic random function of time t and arbitrary random function of coordinates x_1 and x_2. Then the generalized forces $Q_\alpha(t)$ are obviously ergodic stationary functions of time t. Let us introduce the spectral density $\Phi_\alpha(\omega)$ for the generalized forces

$$\Phi_\alpha(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} \overline{Q_\alpha(t)Q_\alpha(t+\tau)} \exp(-i\omega\tau)\, d\tau. \tag{2.4}$$

For mean square generalized forces $\overline{f_\alpha^2}$ we get

$$\overline{f_\alpha^2} = \int_0^{\infty} \frac{\Phi_\alpha(\omega)\, d\omega}{(\omega_\alpha^2 - \omega^2)^2 + (2\beta_\alpha\omega_\alpha\omega)^2}. \tag{2.5}$$

If $\beta_\alpha \ll 1$ and the spectral densities $\Phi_\alpha(\omega)$ are sufficiently slow varying functions of ω, instead of formula (2.5) we can write

$$\overline{f_\alpha^2} \approx \frac{\pi\Phi_\alpha(\omega_\alpha)}{4\beta_\alpha\omega_\alpha^3}. \tag{2.6}$$

Formula (2.6) is the basic relation in the subsequent study.

Let us now assume that the time spectrum of external loads is sufficiently wide. Taking into account the high number of excited modes one can use several simplifications.

(1) The asymptotic expression for the natural modes can be used when the expression for the displacements and stresses in series is introduced. Therefore the asymptotic formulae for generalized forces can be obtained.

(2) The asymptotic formulae for natural frequencies can be used.

(3) Under certain restrictions the summation in formulae for correlation functions, mean squares, etc., may be replaced by integration over wave number space. This permits one to obtain simple integral estimates. The error of these estimates decreases with increasing density of natural frequencies and with decreasing rate of change of parameters in wave number space.

Let the stress $s(t)$ at a certain point, or any other parameter, be expressed in the form

$$s(t) = \sum_\alpha c_\alpha f_\alpha(t), \tag{2.7}$$

where $f_\alpha(t)$ are the generalized coordinates and c_α are certain constants. Then for mean square $\overline{s_\alpha^2}$, neglecting the cross correlation of generalized coordinates and using the formulae (2.6), we obtain an approximate formula

$$\overline{s^2} \approx \sum_\alpha \frac{\pi c_\alpha^2 \Phi_\alpha(\omega_\alpha)}{4\beta_\alpha \omega_\alpha^3}. \tag{2.8}$$

All terms of the series (2.8) depend only on mode number α. In the two-dimensional case this mode is completely defined in asymptotic sense by a pair of wave numbers k_1, k_2. Therefore

$$\overline{s^2} \sim \sum_{k_1}\sum_{k_2} \frac{\pi c^2(k_1, k_2)\Phi(k_1, k_2)}{4\beta(k_1, k_2)\omega^3(k_1, k_2)}, \tag{2.9}$$

where $\Phi(k_1, k_2)$ are the diagonal elements of spectral density matrix for generalized forces. If the terms of series (2.9) are slowly varying functions of wave numbers, the sum on the right side can be replaced by corresponding integral. Then we arrive at

$$\overline{s^2} \sim \frac{\pi}{4\Delta k_1 \Delta k_2} \iint \frac{c^2(k_1, k_2)\Phi(k_1, k_2)\, \mathrm{d}k_1\, \mathrm{d}k_2}{\beta(k_1, k_2)\omega^3(k_1, k_2)}. \tag{2.10}$$

Here Δk_1 and Δk_2 are the dimensions of the quantum cell (Fig. 2). In many cases $\Delta k_1 = \pi/a_1$, $\Delta k_2 = \pi/a_2$. But sometimes, taking into consideration symmetry conditions, we shall put $\Delta k_1 = 2\pi/a_1$, $\Delta k_2 = 2\pi/a_2$, etc. We assume that the integral (2.10) converges and the approximation connected with transition from the formula (2.9) to the formula (2.10) is not too rough. These conditions determine the region of the applicability of the given method.

Formula (2.10) is an asymptotic one. The higher the density of natural frequencies, the smaller is the error in formula (2.10). The error mentioned can be estimated just as in the case of well known summary formulae of the numerical methods of calculations of integrals. The more general expression taking into account the cross correlation, can be developed by the same method. In this case the integral estimate is expressed by means of four-fold integrals.

The integral estimates permit an easy investigation of influence of different factors on a random behaviour of elastic systems. In many problems where standard methods would lead to a great number of calculations, the method of integral estimates leads to simple final formulae. Several applications of this method are considered below.

3. SEVERAL PROBLEMS OF RANDOM VIBRATION OF THIN ELASTIC PLATES

Let us consider the vibrations of a thin elastic rectangular plate with sides a_1 and a_2 and thickness h. The equation for the normal deflection $w(x_1, x_2, t)$ is taken in the form

$$D\Delta\Delta w + \rho h \frac{\partial^2 w}{\partial t^2} + L(w) = q(x_1, x_2, t), \tag{3.1}$$

where L is a linear dissipative operator.

Let us introduce first an operator L_0 given on the set of functions

$$\varphi_\alpha(x_1, x_2, t) = \varphi_\alpha^*(x_1, x_2)\exp(i\omega_\alpha t),$$

where $\varphi_\alpha^*(x_1, x_2)$ is the internal solution (1.7), ω_α is the natural frequency. Here

$$L_0(\varphi_\alpha) = 2i\rho h\beta_\alpha\omega_\alpha\varphi_\alpha^*(x_1, x_2)\exp(i\omega_\alpha t),$$

where β_α is the coefficient of dissipation. Let us consider three realizations of the operator L_0. These realizations are given by the formulae

$$L \doteq \rho h\varepsilon\frac{\partial}{\partial t}, \tag{3.3a}$$

$$L = \frac{\psi_0(D\rho h)^{\frac{1}{2}}}{2\pi}\frac{\partial}{\partial t}\Delta, \tag{3.3b}$$

$$L = \eta D\frac{\partial}{\partial t}\Delta\Delta. \tag{3.3c}$$

The case (a) corresponds to the so called "external" damping with coefficient ε. The case (b) corresponds to the energy dissipation independent of vibration frequency. Here ψ_0 is the specific damping ratio, equal to the rate of energy dissipation during the period of vibration to the mean value of the total energy. The case (c) corresponds to a visco-elastic material according to the Kelvin–Voigt hypothesis with viscosity coefficient η. In the case (a) the damping coefficient β_α is inversely proportional to the frequency ω_α, in the case (b) β_α is independent of ω_α, in the case (c) β_2 is proportional to ω_α.

The next step consists in the extension of these equations to the random processes considered. The extension is based on the asymptotic behaviour of the natural modes. In the asymptotic approximation the natural modes $\varphi_\alpha(x_1, x_2)$ can be approximated by means of $\varphi_\alpha^*(x_1, x_2)$. If the dissipative forces are sufficiently small the evaluation of each generalized coordinate is close to the narrow-band random process with frequency ω_α.

Let the load intensity be delta-correlated in space and possess an arbitrary time correlation. By introducing the Fourier time transformation

$$\Phi_q(x_1, x_2, \xi_1, \xi_2;\ \omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} \overline{q(x_1, x_2, t)q(\xi_1, \xi_2, t+\tau)}\exp(-i\omega\tau)\,d\tau$$

this condition can be written in the form

$$\Phi_q(x_1, x_2, \xi_1, \xi_2; \omega) = \Psi(\omega)a_1a_2\delta(x_1-\xi_1)\delta(x_2-\xi_2). \tag{3.4}$$

Let us substitute formulae (1.5), (1.7) and (3.4) for formula (2.10). Assuming that $c_\alpha = 1$, corresponding to the case of determination of mean square $\overline{w^2}$, we obtain the asymptotic estimate

$$\overline{w^2} \sim \frac{a_1a_2}{4\pi D^2}\left(\frac{D}{\rho h}\right)^{\frac{1}{2}}\iint\frac{\Psi(k_1, k_2)\,dk_1\,dk_2}{\beta(k_1, k_2)(k_1^2+k_2^2)^3}. \tag{3.6}$$

Here the phases x_1^0 and x_2^0 are taken to be uniformly distributed. Therefore $\Delta k_1 = 2\pi/a_1$, $\Delta k_2 = 2\pi/a_2$. The calculations can be simplified by introducing the polar coordinates

$$r^2 = k_1^2 + k_2^2 = \omega\left(\frac{\rho h}{D}\right)^{\frac{1}{2}}, \qquad \operatorname{tg}\theta = k_2/k_1. \tag{3.7}$$

Then the formula (3.6) takes on the form

$$\overline{w^2} \sim \frac{a_1 a_2}{8D^2}\left(\frac{D}{\rho h}\right)^{\frac{1}{2}} \int \frac{\Psi(r)\,\mathrm{d}r}{\beta(r) r}. \tag{3.8}$$

Analogously, for normal stress $\sigma = \sigma_{11}(\pm h/2)$ at arbitrary point of internal region of the plate with coordinates $x_3 = \pm h/2$, we have to take formula (2.10):

$$c(k_1, k_2) = \frac{6D}{h^2}(k_1^2 + \nu k_2^2),$$

where ν is Poisson's ratio. Using notations (3.7), we obtain

$$\overline{\sigma^2} \sim \frac{9a_1 a_2}{16h^4}\left(\frac{D}{\rho h}\right)^{\frac{1}{2}}(3 + 2\nu + 3\nu^2)\int \frac{\Psi(r)\,\mathrm{d}r}{\beta(r) r}. \tag{3.9}$$

Let the time spectrum density be equal to zero everywhere except in the range $\omega_l \leq \omega \leq \omega_u$ inside which the time spectrum density is equal to constant value Ψ. Let us call this white noise truncated from both sides "quasi-white" noise. Thus let us assume

$$\Phi_q(x_1, x_2, \xi_1, \xi_2; \omega) = \begin{cases} \Psi a_1 a_2 \delta(x_1 - \xi_1)\delta(x_2 - \xi_2) \\ (\omega_l \leq \omega \leq \omega_u), \\ 0 \text{ in other cases.} \end{cases} \tag{3.10}$$

Further, let us introduce the characteristic deflection w_0 and characteristic stress σ_0

$$w_0^2 = \frac{a_1 a_2}{8D^2 r_l^2}\left(\frac{D}{\rho h}\right)^{\frac{1}{2}} \frac{\Psi}{\beta_l},$$

$$\sigma_0^2 = \frac{9a_1 a_2}{16h^4}(3 + 2\nu + 3\nu^2)\frac{\Psi}{\beta_l}.$$

Here r_l and β_l are the values of parameters r and β at frequency $\omega = \omega_l$. Using formula (3.8) we obtain the following simple formulae for dimensionless mean square deflection

$$\frac{\overline{w^2}}{w_0^2} \sim \frac{1}{2}\left(1 - \frac{\omega_l}{\omega_u}\right), \tag{3.11a}$$

$$\frac{\overline{w^2}}{w_0^2} \sim \frac{1}{4}\left(1 - \frac{\omega_l^2}{\omega_u^2}\right), \tag{3.11b}$$

$$\frac{\overline{w^2}}{w_0^2} \sim \frac{1}{6}\left(1 - \frac{\omega_l^3}{\omega_u^3}\right). \tag{3.11c}$$

It is interesting that in the limiting case $\omega_u \to \infty$ the mean square displacements are very simply related. The formulae for mean square stresses are

$$\frac{\overline{\sigma^2}}{\sigma_0^2} \sim \frac{1}{2}\left(\frac{\omega_u}{\omega_l}-1\right), \tag{3.12a}$$

$$\frac{\overline{\sigma^2}}{\sigma_0^2} \sim \frac{1}{2}\ln\frac{\omega_u}{\omega_l}, \tag{3.12b}$$

$$\frac{\overline{\sigma^2}}{\sigma_0^2} \sim \frac{1}{2}\left(1-\frac{\omega_l}{\omega_u}\right). \tag{3.12c}$$

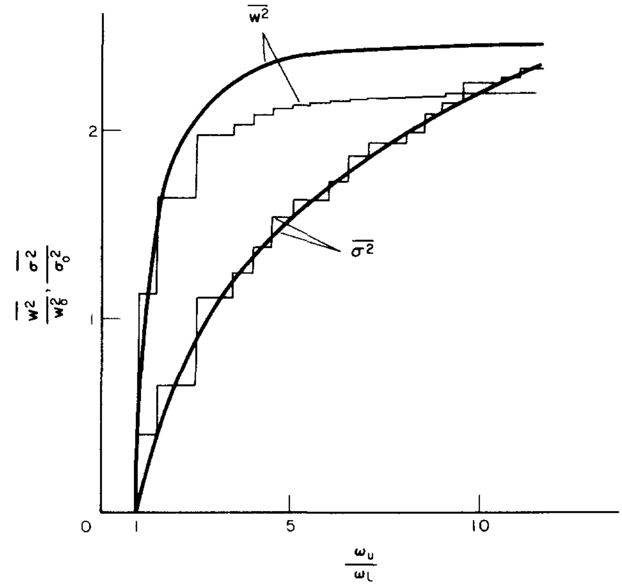

FIG. 3. Comparison of theoretical and empirical values of $\overline{\sigma^2}$ and $\overline{w^2}$.

Let us note that as $\omega_u \to \infty$, the finite limit for $\overline{\sigma^2}$ exists only in case (a). On Fig. 3 the comparison is made between the results obtained by direct summation of terms of series. For calculations we assume $a_1 = a_2 = 1$ m, $h = 2 \,.\, 10^{-3}$ m, $E = 1{\cdot}4 \,.\, 10^{11}$ nm^{-2}, $\nu = 0{\cdot}3$, $\rho = 2{\cdot}7 \,.\, 10^3$ kgm^{-3}, $\omega_l = 3{\cdot}7 \,.\, 10^4$ sec^{-1}. The predicted results are plotted by thick line, the empirical results are presented by means of histogram. The agreement for the stresses is better than the agreement for the displacements. This fact is natural because of the relatively greater contribution of the lowest modes to displacements than to stresses. On the whole the agreement can be considered as satisfactory.

The results obtained above referred to values of parameters in the internal region. Here the influence of boundary conditions was ignored as inessential. It may be expected that the results are applicable for plates of arbitrary form if only the considered points are sufficiently far from the boundary.

Near the contour of the plate substantial boundary effects are to be expected. These effects may be taken into account in the framework of the asymptotic method. First, the mean squares of generalized coordinates are to be calculated under the assumption that the influence of boundary effects on the behaviour of the solution in the internal region is negligible. Then we return to the consideration of boundary conditions introducing in formula (2.10) the corresponding influence coefficients $c(k_1, k_2)$.

As an example let us study the boundary effect near the clamped edges. Using formula (1.6) we find that along the line on which $\sin k_2(x_2 - x_2^0) = 1$

$$c(k_1, k_2) = \frac{72^{\frac{1}{2}}}{h^2} k_1 (k_1^2 + k_2^2)^{\frac{1}{2}}. \tag{3.13}$$

Substituting expression (3.13) into formula (2.10) we obtain after integration

$$\overline{\sigma^2} \sim \frac{9 a_1 a_2}{h^4} \left(\frac{D}{\rho h}\right)^{\frac{1}{2}} \int \frac{\Psi(r)\,\mathrm{d}r}{\beta(r) r}. \tag{3.14}$$

Formulae (3.8) and (3.14) are analogous. The ratio of mean square stresses for clamped edges and for internal region is independent of the time correlation form and type of damping and is equal to

$$\frac{\overline{\sigma^2_{\mathrm{edg}}}}{\overline{\sigma^2_{\mathrm{int}}}} \approx \frac{16}{3 + 2\nu + 3\nu^2}.$$

For $\nu = 0{\cdot}3$ the ratio of the mean square stresses at clamped edges and in the internal region is equal to approximately two.

4. THE INFLUENCE OF LOAD SPACE CORRELATION

The final results given above are obtained for the case when the external forces are delta-correlated. It is the simplest case for the analytical calculation but it is a too rough an idealization of a real process. The question concerning the possibilities of such idealization can be studied also by means of the method presented. For simplicity let us consider the one-dimensional system—a beam or a plate of span a subjected to cylindrical bending. Let the external forces be exponentionally correlated. Then

$$\Phi_q(x, \xi; \omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} \overline{q(x, t) q(\xi, t + \tau)} e^{i\omega\tau} = \Psi(\omega) e^{-\alpha|x - \xi|}. \tag{4.1}$$

Here α is the correlation constant, $\Psi(\omega)$ is the function characterizing the time correlation. The spectral density of generalized forces can be determinated by formulae (2.3) and (2.4). Expressing $\Phi_\alpha(\omega)$ as a function of wave number k we obtain

$$\Phi(k) = \frac{4\Psi(k)}{\rho^2 h^2 a} \left[\frac{\alpha a}{k^2 + \alpha^2} + \frac{2k^2}{(k^2 + \alpha^2)^2} (1 + e^{-\alpha a})\right]. \tag{4.2}$$

Let us substitute the expressions (1.5) and (4.2) into formula (2.10). As a result we obtain an asymptotic formula for mean square displacements and stresses in the internal region ($x_3 = \pm h/2$):

$$\overline{w^2} \sim \frac{1}{2D^2 a}\left(\frac{D}{\rho h}\right)^{\frac{1}{2}} \int \left[\alpha a + \frac{2k^2}{k^2+\alpha^2}(1+e^{-\alpha a})\right] \frac{\Psi(k)\,\mathrm{d}k}{\beta(k)(k^2+\alpha^2)k^6}, \tag{4.3}$$

$$\overline{\sigma^2} \sim \frac{18}{ah^4}\left(\frac{D}{\rho h}\right)^{\frac{1}{2}} \int \left[\alpha a + \frac{2k^2}{k^2+\alpha^2}(1+e^{-\alpha a})\right] \frac{\Psi(k)\,\mathrm{d}k}{\beta(k)(k^2+\alpha^2)k^2}. \tag{4.4}$$

As an example let us consider the case of "quasiwhite" time noise. Let the damping be external (case a). Then $\beta(k) = \beta_l k_l^2/k^2$, where k_l and β_l are the wave number and damping coefficient, respectively, corresponding to the lowest frequency ω_l. Introducing the notations

$$\sigma_0^2 = \frac{18}{h^4}\left(\frac{D}{\rho h}\right)^{\frac{1}{2}} \frac{\Psi}{\beta_l k_l^2}, \qquad \mu_\alpha = \alpha a, \qquad \mu_l = k_l a, \qquad \mu_u = k_u a$$

after calculation using formula (4.4) we obtain

$$\frac{\overline{\sigma^2}}{\sigma_0^2} \sim (1-e^{-\mu_\alpha})\left(\frac{\mu_l}{\mu_l^2+\mu_\alpha^2} - \frac{\mu_u}{\mu_u^2+\mu_\alpha^2}\right) + \frac{1+e^{-\mu_\alpha}+\mu_\alpha}{\mu_\alpha}\left(\operatorname{arc\,tg}\frac{\mu_u}{\mu_\alpha} - \operatorname{arc\,tg}\frac{\mu_l}{\mu_\alpha}\right). \tag{4.5}$$

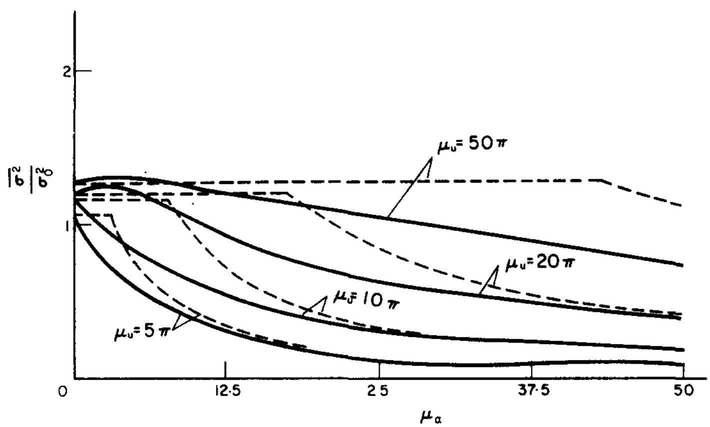

FIG. 4. Dependence of $\overline{\sigma^2}$ on μ_α and μ_u.

The results of calculations by formula (4.5) are plotted on Fig. 4 by continuous lines. It was assumed that $\mu_l = \pi$. The horizontal dotted lines correspond to the assumption that the load is completely space correlated. The dotted curves are obtained for space delta-correlated load. The load intensity is taken so that at great values of α the exponentially-correlated and delta-correlated loads are equivalent in a certain sense. To this end instead of formula (4.1) the expression

$$\Phi_q(x, \xi; \omega) = \frac{2}{\alpha}\Psi(\omega)\delta(x-\xi)$$

was taken. From the diagram (Fig. 4) it follows that the wider the time spectrum of excitation the smaller must be the space correlations. This follows also from mechanical considerations. An unexpected result is that the complete correlation approximation is valid over a very wide range of damping coefficient variation, provided the time spectrum of excitation is sufficiently wide. This result should be studied further, as also other questions concerning the possibilities of simple analytical approximation of realistic random loadings.

5. THE INFLUENCE OF TRANSVERSE SHEAR AND OF ROTATIONARY INERTIA

At sufficiently high wave numbers the effects of transverse shear and rotationary inertia must be taken into account. Let us estimate the influence of these factors on the mean square stresses and displacements. Consider a rectangular plate with sides a_1 and a_2 and with thickness h. Let us take the equations which follow from the Reissner theory [10]:

$$\begin{aligned}
&\Delta\varphi - \frac{6\gamma(1-\nu)}{h^2}(\varphi - w) - \frac{\rho(1-\nu^2)}{E}\frac{\partial^2\varphi}{\partial t^2} = 0,\\
&\Delta\psi - \frac{12\gamma}{h^2}\psi - \frac{2\rho(1+\nu)}{E}\frac{\partial^2\psi}{\partial t^2} = 0, \qquad (5.1)\\
&\Delta(w-\varphi) - \frac{2\rho(1+\nu)}{\gamma E}\frac{\partial^2 w}{\partial t^2} = q.
\end{aligned}$$

Here $w(x_1, x_2, t)$ is the normal displacement of the points of the middle surface, $\varphi(x_1, x_2, t)$ and $\psi(x_1, x_2, t)$ are the displacement potentials which are connected to the angles of rotation of normals θ_1 and θ_2 by expressions

$$\theta_1 = \frac{\partial\varphi}{\partial x_1} + \frac{\partial\psi}{\partial x_2}, \qquad \theta_2 = \frac{\partial\varphi}{\partial x_2} - \frac{\partial\psi}{\partial x_1}.$$

Further, E is the elastic modulus, ν is Poisson's ratio, ρ is the density, γ is a nondimensional coefficient, taken usually equal to 5/6.

It was shown by Moskalenko [11] that the asymptotic method is valid for the first, primarily (bending) series of the natural modes if the characteristic wave length is greater than the thickness h. More exactly, the condition of the application of the asymptotic method is

$$r^2 = k_1^2 + k_2^2 < \frac{\pi^2}{h^2}, \qquad (5.2)$$

where k_1 and k_2 are the wave numbers. For the natural vibration frequencies, corresponding to the first (bending) series, we obtain the approximate formula

$$\omega^2 \approx \frac{D}{\rho h}\frac{r^4}{1 + \varkappa h^2 r^2}. \qquad (5.3)$$

Here $\varkappa$ is the nondimensional coefficient

$$\varkappa = \frac{1}{6\gamma(1-\nu)} + \frac{1}{12}. \tag{5.4}$$

Using the method given above we obtain the formula for mean square displacements in the internal region of plate under the loading with the spectral density (3.4):

$$\overline{w^2} \sim \frac{a_1 a_2}{8D^2}\left(\frac{D}{\rho h}\right)^{\frac{1}{2}} \int \frac{\Psi(r)(1+\varkappa h^2 r^2)^{\frac{1}{2}}\,dr}{\beta(r) r^5}. \tag{5.5}$$

For mean square stress $\sigma = \sigma_{11}(\pm h/2)$ we obtain analogously in the internal region

$$\overline{\sigma^2} \sim \frac{9a_1 a_2}{16h^4}\left(\frac{D}{\rho h}\right)^{\frac{1}{2}}(3+2\nu+\nu^2) \int \frac{\Psi(r)(1+\varkappa h^2 r^2)^{\frac{1}{2}}\,dr}{\beta(r)(1+\varkappa_0 h^2 r^2)^2 r}. \tag{5.6}$$

Here $\varkappa_0 = \varkappa - 1/12$, where $\varkappa$ is determined by formula (5.4). Let us carry out the calculations by formula (5.6) for the case of "quasiwhite noise". Further, let us take $\beta = \text{const}$. The calculation by formula (5.6) gives

$$\begin{aligned}\frac{\overline{\sigma^2}}{\sigma_0^2} \sim \ln \frac{r_u[1+(1+\varkappa h^2 r_l^2)^{\frac{1}{2}}]}{r_l[1+(1+\varkappa h^2 r_u^2)^{\frac{1}{2}}]} &+ \frac{1}{2}\left(1-\frac{\varkappa}{\varkappa_0}\right)\left[\frac{(1+\varkappa h^2 r_u^2)^{\frac{1}{2}}}{1+\varkappa_0 h^2 r_u^2} - \frac{(1+\varkappa h^2 r_l^2)^{\frac{1}{2}}}{1+\varkappa_0 h^2 r_l^2}\right] \\ &+ \frac{1}{2}\left(\frac{\varkappa}{\varkappa+\varkappa_0}\right)^{\frac{1}{2}}\left(\frac{\varkappa}{\varkappa_0}+\frac{\varkappa^2}{\varkappa_0^2}-2\right)\left\{\operatorname{arc\,tg}\left[\frac{\varkappa_0(1+\varkappa h^2 r_u^2)}{\varkappa-\varkappa_0}\right]^{\frac{1}{2}} - \operatorname{arc\,tg}\left[\frac{\varkappa_0(1+\varkappa h^2 r_l^2)}{\varkappa-\varkappa_0}\right]^{\frac{1}{2}}\right\}.\end{aligned} \tag{5.7}$$

In this formula σ_0 is the characteristic stress

$$\sigma_0^2 = \frac{9a_1 a_2}{16h^4}\left(\frac{D}{\rho h}\right)^{\frac{1}{2}}(3+2\nu+3\nu^2)\frac{\Psi}{\beta}.$$

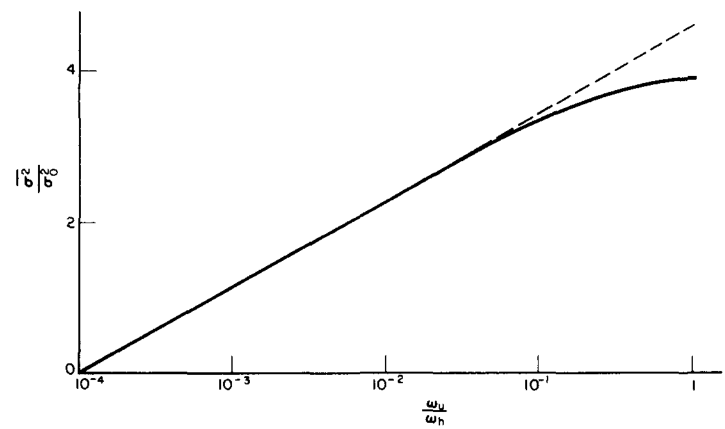

FIG. 5. Dependence of $\overline{\sigma^2}$ on ω_u/ω_h for a thick plate.

On Fig. 5 the diagram corresponding to formula (4.7) is given. Here ω_h is a characteristic frequency

$$\omega_h = \frac{1}{\varkappa h^2}\left(\frac{D}{\rho h}\right)^{\frac{1}{2}},$$

The order of magnitude of ω_h is equal to natural frequency corresponding to the wavelength of the order of thickness h. In plotting the diagram it was assumed that the lower boundary of the exciting spectrum is $\omega_l = 10^{-4}\omega_h$. The result corresponding to the classical theory is plotted by a dotted line. It is seen from the diagram that the classical theory of plates is applicable up to $\omega_u \sim 10^{-1}\omega_h$. As an approximate estimate we can use the classical theory practically up to $\omega_u \sim \omega_h$.

6. RANDOM VIBRATION PROBLEMS FOR THIN ELASTIC SHELLS

Let us consider now the more difficult problem of random vibration of thin shells. Let us use the system of differential equations

$$\begin{gathered} D\Delta\Delta w - \left(\varkappa_2 \frac{\partial^2 \psi}{\partial x_1^2} + \varkappa_1 \frac{\partial^2 \psi}{\partial x_2^2}\right) + \rho h \frac{\partial^2 w}{\partial t^2} = q, \\ \frac{1}{Eh}\Delta\Delta\psi + \varkappa_2 \frac{\partial^2 w}{\partial x_1^2} + \varkappa_1 \frac{\partial^2 w}{\partial x_2^2} = 0, \end{gathered} \tag{6.1}$$

describing vibrations with sufficiently high wave numbers and ignoring the tangential external forces and tangential inertia forces. In the equations (6.1) the same notation is used as in equations (1.1). However, in equations (6.1) $w(x_1, x_2, t)$ and $\psi(x_1, x_2, t)$ are the functions describing the time variation of the displacements and stresses, respectively.

Using the integral method we can obtain the formula for mean square displacements and stresses, which coincides with formula (2.10). The expression (1.2) must be substituted into the formula (2.10) instead of $\omega(k_1, k_2)$. Let the external load be characterized by expression (3.4). After substitution and transition to polar coordinates (3.7), formula (2.10) takes on the form

$$\overline{w_2} \sim \frac{a_1 a_2}{4\pi D}\left(\frac{D}{\rho h}\right)^{\frac{1}{2}} \int_0^{\pi/2} d\theta \int_{r_l(\theta)}^{r_u(\theta)} \frac{\Psi(r, \theta) r \, dr}{[r^4 + r_0^4(\chi \cos 2\theta + \sin^2\theta)^2]^{\frac{3}{2}}}. \tag{6.2}$$

In formula (6.2) r_0 indicates the constant

$$r_0^4 = \frac{12(1-\nu^2)\varkappa_1^2}{h^2} \tag{6.3}$$

and $r_l(\theta)$ and $r_u(\theta)$ indicate the real roots of the equation

$$r^4 + r_0^4(\chi \cos^2\theta + \sin^2\theta)^2 = \omega^2$$

corresponding to the cases $\omega = \omega_l$ and $\omega = \omega_u$.

The integral on the right side of formula (6.2), in general, is not expressible by elementary functions. However, in some cases, as an example in the case of a spherical shell, simple final formulae can be obtained. Let Ψ = const in the interval $\omega_l \leq \omega \leq \omega_u$

and be equal to zero outside this interval. Here $\omega_l \geq \omega_R$, where ω_R is the minimum natural frequency determined by formula

$$\omega_R = \frac{1}{R}\left(\frac{E}{\rho}\right)^{\frac{1}{2}}. \tag{6.4}$$

Introducing the characteristic deflection w_0 by formula

$$w_0^2 = \frac{a_1 a_2}{8D^2 r_l^4}\left(\frac{D}{\rho h}\right)^{\frac{1}{2}} \frac{\Psi}{\beta_l}$$

we obtain after integration

$$\frac{\overline{w^2}}{w_0^2} \sim \frac{1}{2}\frac{\omega_l}{\omega_R}\left[\operatorname{arc\,tg}\left(\frac{\omega_u^2}{\omega_R^2}-1\right)^{\frac{1}{2}} - \operatorname{arc\,tg}\left(\frac{\omega_l^2}{\omega_R^2}-1\right)^{\frac{1}{2}}\right], \tag{6.5a}$$

$$\frac{\overline{w^2}}{w_0^2} \sim \frac{1}{2}\frac{\omega_l^2}{\omega_R^2}\left[\left(1-\frac{\omega_R^2}{\omega_u^2}\right)^{\frac{1}{2}} - \left(1-\frac{\omega_R^2}{\omega_l^2}\right)^{\frac{1}{2}}\right], \tag{6.5b}$$

$$\begin{aligned}\frac{\overline{w^2}}{w_0^2} \sim \frac{1}{4}\frac{\omega_l^3}{\omega_R^3}\Bigg[&\frac{\omega_R^2}{\omega_u^2}\left(\frac{\omega_u^2}{\omega_R^2}-1\right)^{\frac{1}{2}} - \frac{\omega_R^2}{\omega_l^2}\left(\frac{\omega_l^2}{\omega_R^2}-1\right)^{\frac{1}{2}} \\ &-\operatorname{arc\,tg}\left(\frac{\omega_u^2}{\omega_R^2}-1\right)^{\frac{1}{2}} + \operatorname{arc\,tg}\left(\frac{\omega_l^2}{\omega_R^2}-1\right)^{\frac{1}{2}}\Bigg].\end{aligned} \tag{6.5c}$$

As before, the first formula corresponds to the case of external damping, the second one corresponds to the case of damping independent of the frequency, the third formula corresponds to the case of Kelvin–Voigt damping.

Some numerical results obtained for case a are plotted on Fig. 6. It is seen from the diagram that the influence of the curvature on the mean square displacements is not too strong. It is seen from Fig. 6 that the mean square deflections increase when the ratio

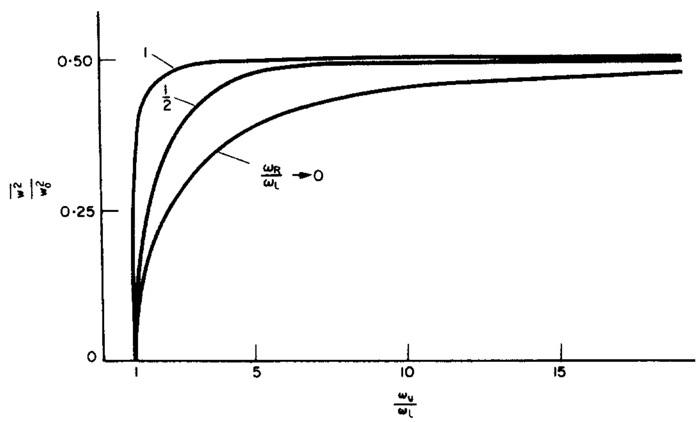

FIG. 6. Dependence of $\overline{w^2}$ on ω_u and ω_l for a spherical shell.

ω_R/ω_l approaches unity. This may be explained by the approximation of the lower part of the interval of exciting frequencies to the frequency near which a maximum of the density of natural frequencies takes place [9].

For the calculation of mean square normal stresses $\sigma = \sigma_{11}(\pm h/2)$ at points located sufficiently far from the boundary, we have to take in formula (2.10)

$$c(k_1, k_2) = \frac{6D}{h^2}\left[k_1^2 + \nu k_2^2 + \frac{Eh^2|\varkappa_1|}{6D}\,\frac{(k_1^2\chi + k_2^2)k_2^2}{k_1^2 + k_2^2}\right].$$

The substitution into the formula yields

$$\overline{\sigma^2} \sim \frac{9a_1a_2}{\pi h^4}\left(\frac{D}{\rho h}\right)^{\frac{1}{2}}\int_0^{\pi/2} d\theta \int_{r_l(\theta)}^{r_u(\theta)} \frac{\Psi(r,\theta)[r^2(\cos^2\theta + \nu\sin^2\theta + \alpha r_0^4(\chi\cos^2\theta + \sin^2\theta)\sin^2\theta]^2 r\,dr}{\beta(r,\theta)[r^4 + r_0^4(\chi\cos^2\theta + \sin^2\theta)^2]^{\frac{3}{2}}}, \tag{6.6}$$

where $\alpha^2 = (1-\nu^2)/3$.

As an example, let us consider the spherical shell loaded by the broadband random time–space forces (3.10). Let the damping be external (case a). Introducing the notation for the characteristic stress

$$\sigma_0^2 = \frac{9a_1a_2}{16h^4}\left(\frac{D}{\rho h}\right)^{\frac{1}{2}}\frac{\Psi}{\beta_l}(3 + 2\nu + 3\nu^2)$$

we obtain after the calculation of the integral in formula (6.6)

$$\frac{\overline{\sigma^2}}{\sigma_0^2} \sim \frac{1}{2}\frac{\omega_R}{\omega_l}\left\{\left(\frac{\omega_u^2}{\omega_R^2} - 1\right)^{\frac{1}{2}} - \left(\frac{\omega_l^2}{\omega_R^2} - 1\right)^{\frac{1}{2}} + \frac{(1+3\nu)\varkappa}{3+2\nu+3\nu^2}\ln\frac{\omega_u^2}{\omega_R^2} - \frac{3+4\nu+\nu^2}{3+2\nu+3\nu^2}\left[\operatorname{arc\,tg}\left(\frac{\omega_u^2}{\omega_R^2} - 1\right)^{\frac{1}{2}} - \operatorname{arc\,tg}\left(\frac{\omega_l^2}{\omega_R^2} - 1\right)^{\frac{1}{2}}\right]\right\}. \tag{6.7}$$

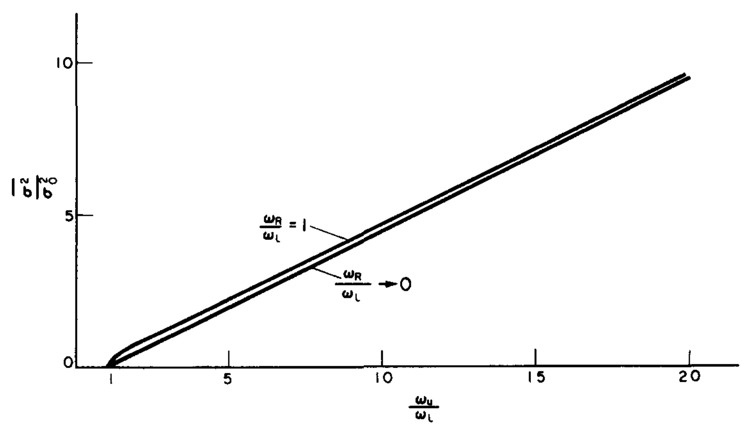

FIG. 7. Dependence of $\overline{\sigma^2}$ on ω_u and ω_l for a spherical shell.

The results of calculation by formula (6.7) are plotted on Fig. 7. Taking into account the asymptotic character of the used estimate, the influence of the curvature on the mean square stress may be neglected (if the exciting spectrum is sufficiently wide).

7. ASYMPTOTIC ESTIMATES FOR SPECTRAL DENSITIES AND OTHER CHARACTERISTICS OF FREQUENCIES

The asymptotic method can be used for calculation of spectral densities, correlation functions, etc. As an example let us calculate the spectral density of stresses at some point of a plate or a shell. The connection between the mean square $\overline{\sigma^2}$ and spectral density $\Phi_\sigma(\omega)$ is given by formula

$$\overline{\sigma^2} = \int_0^\infty \Phi_\sigma(\omega)\,\mathrm{d}\omega. \tag{7.1}$$

Together with the given excitation process let us consider an auxiliary process with spectral density coinciding with the spectral density of the given process at $\omega \leq \omega_*$ and being equal to zero at $\omega > \omega_*$. Using the method described above, we may obtain an asymptotic estimate for mean square $\overline{\sigma^2} \sim f_\sigma(\omega_*)$ as a function of separation frequency ω_*. Then the asymptotic estimate for spectral density of stresses is expressed on the basis of (7.1) by the simple formula

$$\Phi_\sigma(\omega) \sim \left.\frac{\partial f_\sigma(\omega_*)}{\partial \omega_*}\right|_{\omega=\omega_*}. \tag{7.2}$$

As an example, let us carry out the determination of the spectral density of stresses near the clamped edge of a thin plate. Let us consider the loading given in the form (3.10). Analogously to formula (3.14) let us write

$$f_\sigma(\omega_*) \sim \frac{9a_1a_2}{h^4}\left(\frac{D}{\rho h}\right)^{\frac{1}{2}} \int_0^{\omega_*^{1/2}(\rho h/D)^{1/4}} \frac{\Psi(r)\,\mathrm{d}r}{\beta(r)r}.$$

Hence, using formula (7.2), we obtain an estimate for the spectral density

$$\Phi_\sigma(\omega) \sim \frac{9a_1a_2}{2h^4}\left(\frac{D}{\rho h}\right)^{\frac{1}{4}} \frac{\Psi(\omega)}{\beta(\omega)\omega}. \tag{7.3}$$

Let us note that this estimate is an asymptotical one. The spectral density calculated by means of formula (7.3) is smoothed out with respect to the peaks due to the individual contribution of each natural mode. It is evident that the error in estimating the spectral densities may be larger than in estimating the mean squares. However, the formulae of type of (7.3) may be useful for drawing general conclusions concerning the influence of different factors on the character of the spectral density. On Fig. 8 the diagrams for smoothed spectral density of deflection are plotted for the case when the loading is space white noise and time quasiwhite noise, i.e. the time–space spectrum is given by expression (3.10). The variation of the character of spectral density in the transition from the external damping (case a) to Voigt's damping (case c) is shown on the diagram. An analogous diagram for the spectral density of stresses is given on Fig. 9. In the case of external damping

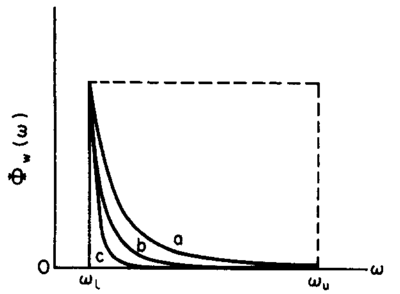

FIG. 8. Spectral density $\Phi_w(\omega)$ for three cases of dissipation.

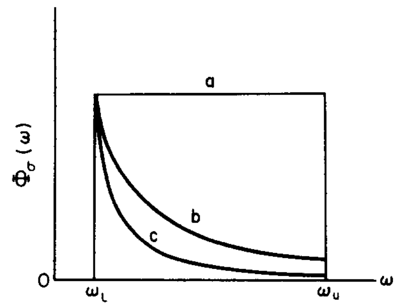

FIG. 9. Spectral density $\Phi_\sigma(\omega)$ for three cases of dissipation.

the spectral density of stresses is similar to the time spectral density of load. In the case of damping independent of frequency and in the case of Voigt's damping, the higher natural modes are suppressed more than the lower natural modes.

It is not difficult to obtain the asymptotic estimates for some other values. Thus in determining the degree of damage and the mean life of structures [8] we must know the effective frequency ω_{eff} of stresses $\sigma(t)$:

$$\omega_{\text{eff}} = \left[\frac{\int_0^\infty \omega^2 \Phi_\sigma(\omega)\, d\omega}{\int_0^\infty \Phi_\sigma(\omega)\, d\omega}\right]^{\frac{1}{2}}. \tag{7.4}$$

Using the asymptotic estimate (7.3) for the spectral density, we obtain the following simple formulae for ω_{eff}:

$$\omega_{\text{eff}}^2 \sim \frac{\omega_l^2 + \omega_u\omega_l + \omega_u^2}{3}, \tag{7.5a}$$

$$\omega_{\text{eff}}^2 \sim \frac{\omega_u^2 - \omega_l^2}{\ln \dfrac{\omega_u^2}{\omega_l^2}}, \tag{7.5b}$$

$$\omega_{\text{eff}}^2 \sim \omega_u\omega_l. \tag{7.5c}$$

The formulae (7.5) indicate the influence of the time spectrum width and type of damping on the mean life of the structure.

REFERENCES

[1] S. H. CRANDALL, *Random Vibration*, Technology Press, Mass. Inst. Techn., (1958).

[2] V. V. BOLOTIN, Boundary effect in vibrations of elastic shells (Kraevoi effekt pri kolebaniiakh uprugikh obolochek), *Prikl. Math. i. Mekh.*, **24**, No. 5 (1960).

[3] V. V. BOLOTIN, Asymptotic method for investigating eigenvalue problems for rectangular regions. (Asimptoticheskii metod issledovaniia zadach o sobstvennykh znacheniiakh dlia priamougol'nykh oblastei), *Problems of mechanics of continuous media* (anniversary volume of N. I. Muskhelishvili) (*Problemy mekhaniki sploshnoi sredy*) (Iubileinyi sbornik N. I. Muskhelishvili)), AN SSSR (1961).

[4] V. V. BOLOTIN, On elastic vibrations excited by random forces with a broad spectrum (Ob uprugikh kolebaniiakh, vozbuzhdaemykh sluchainymi silami s shirokim spektrom), Izvestiia vysshikh uchebnykh zavedenii, Mashinostroenie, No. 4 (1963).

[5] V. V. BOLOTIN, On the broadband random vibration of elastic systems, *Proc. of the Eleventh International Congress of Applied Mechanics*, Springer-Verlag, (1965).
[6] S. IGUCHI, Die Biegungsschwingungen der vierseitig eingespannten rechteckigen Platten. *Ing.-Arch.*, **8**, No. 1, (1937).
[7] R. COURANT and D. HILBERT, *Methods of Mathematical Physics*, Vol. 1, Gostekhizdat (1951).
[8] V. V. BOLOTIN, *Statistical methods in structural mechanics* (*Statisticheskie metody v stroitel' noi mekhanike*), Stroizdat (1961).
[9] V. V. BOLOTIN, On the density of frequencies of free vibrations of thin elastic shells (O plotnosti chastot sobstvennykh kólebanii tonkikh uprugikh obolochek), *Prikl. Math i Mekh.*, **27**, No. 2 (1963).
[10] E. REISSNER, On Bending of Elastic Plates, *Quart. Appl. Math.*, **5**, No. 1 (1947).
[11] V. N. MOSKALENKO, On the application of more accurate equations of plate bending to the problem of free vibrations (K primeneniiu utochnennykh uravnenii izgiba plastinok k zadache o sobstvennykh kolebaniiakh), *Inzhenernyi zh.*, No. 3 (1961).

(*Received* 10 *May* 1965)

Résumé—Dans ce texte sont considérées les vibrations des systèmes élastiques sous l'action de forces externes, ce qui représente un procédé au hasard espace-temps à large bande. Des évaluations intégrales pour les déplacements et efforts qui adviennent dans les systèmes élastiques ont été proposées en se basant sur la méthode asymptotique avancée par l'auteur et sur des évaluations intégrales pour la densité des fréquences naturelles d'une voile. A la base de la méthode avancée se trouve le remplacement de la sommation de modes naturels séparés de vibrations libres par une intégration d'une certaine gamme de fréquences (ou numéros d'ondes). Le problème des limites de cette méthode y est discuté. L'influence des caractéristiques de charge et du genre d'amortissement ainsi que du genre des conditions limites sur les carrés moyens et densités spectrales des déplacements et efforts, est investiguée.

Zusammenfassung—Es werden Schwingungen von elastischen Systemen unter der Einwirkung von äusseren Kräften untersucht, welche einen zufälligen Breitband Raum-Zeitvorgang darstellen. Auf Grund der vom Verfasser vorgeschlagenen asymptotischen Methode und der Schätzungen der Dichte von Eigenfrequenzen von Schalen, werden integrale Schätzungen für Verschiebungen und Spannungen vorgeschlagen, welche in elastischen Systemen auftreten. Am Grunde der vorgeschlagenen Methode liegt der Ersatz der Summierung über einzelne Hauptformen der Eigenschwingungen durch Integration über einen gewissen Frequenzbereich (oder Bereich der Wellenzahlen). Die Frage betreffend die Einschränkungen dieser Methode wird erörtert. Untersucht wird ferner der Einfluss der Belastungscharakteristiken, der Dämpfungsart, als auch der Art der Randbedingungen auf die Durchschnittsquadrate und auf die Spektraldichten der Verschiebungen und Spannungen.

Аннотация—Рассматриваются колебания упругих систем под действием внешних сил, представляющих собой пространственно-временной сиучайный процесс с широким спектром. На основе асимптотического метода, предложенного автором, и оценок для плотности собственных частот оболочек предлагаются интегральные оценки для перемещений и напряжений, возникающих в упругих системах. В основе применяемого метода лежит замена суммирования по отдельным формам собственных колебаний интегрированием по некоторой области частот (или волновых чисел). Обсуждается вопрос о границах применения этого метода. Исследуется влияние характеристик нагрузки, типа демпфирования, а также типа граничных условий на средние квадраты и спектральные плотности перемещений и напряжений.